from these measured quantities.

$$\langle t_d\rangle(N_1-N_2)=N_1\langle t_1\rangle-N_2\langle t_2\rangle. \quad \text{(A2)}$$

The mean life $\tau$, characteristic of the decay process, is then obtained as a solution of Eq. (A1).

Let $\delta$ be the standard statistical uncertainty on $\langle t_d\rangle$. This may be expressed directly in terms of measured quantities, viz;

$$\delta^2(N_1-N_2)^2 = N_1[\langle(t_1)^2\rangle-(\langle t_1\rangle)^2+(\langle t_1\rangle-\langle t_d\rangle)^2] + N_2[\langle(t_2)^2\rangle-(\langle t_2\rangle)^2+(\langle t_2\rangle-\langle t_d\rangle)^2], \quad \text{(A3)}$$

where $\langle(t_i)^2\rangle\equiv$ the mean of the squared lifetime of *all* events observed during the $i$th interval $T$.

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# Regge Poles and the Photoproduction of Pions*

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The photoproduction of pions, $\gamma+N\rightarrow\pi+N$, is examined from the standpoint of the Regge pole hypothesis and the Mandelstam representation. The asymptotic behavior of the forward scattering amplitudes is determined in terms of the Regge trajectories of the $\gamma+\pi\rightarrow N+\bar{N}$ channel. The $\rho$, $\phi$, and $\omega$ trajectories are included in the description of the photoproduction of neutral pions, whereas only the $\rho$ and $\pi$ trajectories contribute to the photoproduction of charged pions. In the case of backward scattering, asymptotic representations of the scattering amplitudes are controlled by the trajectories of the $\gamma+N\rightarrow\pi+N$ channel by crossing. Finally, generalized Pomeranchuk relations are established for the differential cross sections in the forward and backward directions for the various charge configurations of the photoproduction channel. In particular, we have the following interesting results: (1) The differential cross sections for $\gamma+p\rightarrow n+\pi^+$ and $\gamma+n\rightarrow p+\pi^-$ are asymptotically equal in the forward and backward directions; (2) the differential cross sections for $\gamma+p\rightarrow p+\pi^0$ and $\gamma+n\rightarrow n+\pi^0$ are asymptotically equal in the backward direction.

## I. INTRODUCTION

WITH the exception of selection rules, the underlying physical principles of elementary-particle physics are currently being expressed in terms of analyticity properties of transition amplitudes in a manner consistent with unitarity. In fact, for strongly interacting systems, the principle of maximal analyticity in linear momentum has been frequently invoked.[1] The resulting description, i.e., the Mandelstam representation and unitarity, is incomplete at least to the extent that the behavior of the scattering amplitude at infinity remains undetermined. Complex angular momentum may be useful in this respect since Regge[2] has shown that the meromorphicity and asymptotic boundedness of the partial-wave amplitudes continued to complex $J$ provide boundary conditions for the scattering amplitude at infinity. Although Regge's work is for potential scattering and relativistic proofs of certain aspects of this program are still lacking, it is desirable, nevertheless, to investigate the consequences of this approach since the equations obtained for the scattering amplitudes are simple in form and are subject to experimental verification.

These and other considerations have led us to examine a process in which both strong and electromagnetic interactions enter, namely the photoproduction of pions. The basic assumption is that the particles mediating the strong interaction correspond to certain Regge trajectories in the complex $J$ plane.

Our main purpose is to demonstrate the existence of generalized Pomeranchuk relations for the photoproduction of pions. These relations do not necessarily pertain to total cross sections of particle and antiparticle reactions, as did the original Pomeranchuk theorems. Instead, it is recognized that the fundamental mechanism responsible for the Pomeranchuk theorems, namely a dominant Regge trajectory, may also be the author of other asymptotic symmetries. Wagner and Sharp,[3,4] for instance, have discussed such asymptotic relationships between the differential cross sections for the direct and crossed channels of several reactions. For photoproduction the equality of the differential cross sections for the direct and crossed channels is guaranteed at all energies by invariance under charge conjugation. We, therefore, turn to the particular charge configurations present in a given channel, and it is found that they satisfy asymptotic symmetries of this type.

The general plan of the paper is as follows. In Sec. II, the basic kinematics are outlined and the amplitudes

* Supported in part by the National Science Foundation.
[1] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).
[2] T. Regge, Nuovo Cimento **14**, 951 (1959).
[3] W. G. Wagner and D. H. Sharp, Phys. Rev. **128**, 2899 (1962).
[4] W. G. Wagner, Phys. Rev. Letters **10**, 202 (1963).

satisfying the Mandelstam representation are introduced and related to the angular momentum expansions of channel II, $\gamma+\pi\rightarrow N+\bar{N}$. The notation and results of Ball[5] are used extensively. In Sec. III, we summarize the approach that allows a description in terms of complex angular momenta. In Sec. IV, we discuss the poles of the eigenamplitudes for channel II. Asymptotic expressions for the scattering amplitudes are obtained in Sec. V.

## II. KINEMATICS

Associated with this problem are three reactions:

$$\gamma+N_1\rightarrow\pi+N_2;$$

$$\gamma+\pi\rightarrow\bar{N}_1+N_2;$$

and

$$\gamma+\bar{N}_2\rightarrow\pi+\bar{N}_1,$$

which we designate as channels I, II, and III, respectively. Since channel III is the charge conjugate of channel I, if the subscripts are interchanged, it is not discussed in detail in this section. Conservation of four momentum in channel I is written as

$$K+P_1=Q+P_2, \qquad (2.1)$$

where $K$, $P_1$, $Q$, and $P_2$ refer to the $\gamma$, $N_1$, $\pi$, and $N_2$, respectively. In the barycentric system we have the relations, $K=(\mathbf{K},k)$, $P_1=(-\mathbf{K}, E_1=[M^2+k^2]^{1/2})$, $Q=(\mathbf{Q},\omega=[q^2+1]^{1/2})$, and $P_2=(-\mathbf{Q}, E_2=[q^2+M^2]^{1/2})$, where $k=|\mathbf{K}|$ and $q=|\mathbf{Q}|$. For channel II, $P_1$ and $Q$ have negative components so that $P_1'=-P_1$ and $Q'=-Q$ refer to the outgoing antinucleon and the incoming pion, respectively. The constraint expressed by Eq. (2.1) can be written in terms of the physical variables of channel II as

$$K+Q'=P_1'+P_2, \qquad (2.2)$$

where $K=(\mathbf{K},k')$, $Q'=(-\mathbf{K},\omega=[k'^2+1]^{1/2})$, $P_1'=(-\mathbf{P}_2, E=[p^2+M^2]^{1/2})$, $P_2=(\mathbf{P}_2, E=[p^2+M^2]^{1/2})$, and $k'$, $p$ are the magnitudes of $\mathbf{K}$, $\mathbf{P}_2$, respectively.

We define a transition matrix element $T$ in channel I by the equation

$$S=i(2\pi)^{-2}\delta(K+P_1-Q-P_2)\times M(4E_1E_2k\omega)^{-1/2}\bar{u}(P_2)Tu(P_1) \qquad (2.3a)$$

and in channel II by the equation

$$S=i(2\pi)^{-2}\delta(K+Q'-P_1'-P_2)\times M(4E^2k'\omega)^{-1/2}\bar{u}(P_2)Tv(P_1'), \qquad (2.3b)$$

where $u(P_2)$ is a positive energy Dirac spinor, $v(P_1')$ is a charge conjugate spinor, $v(P_1')=Cu^{-T}(P_1')$, $C\gamma_u{}^TC^{-1}=-\gamma_u$, and $C^T=-C$. Chew, Goldberger, Low, and Nambu[6] have shown that the most general form for $T$ is given by

$$T_\beta=g_\beta{}^{(+)}T^{(+)}+g_\beta{}^{(-)}T^{(-)}+g_\beta{}^{(0)}T^{(0)} \qquad (2.4a)$$

and

$$T^{(+,-,0)}=\sum_{n=1}^{4}O_nB_n{}^{(+,-,0)}, \qquad (2.4b)$$

where $g_\beta{}^{(+)}=\delta_{\beta 3}$, $g_\beta{}^{(-)}=\frac{1}{2}[\tau_\beta,\tau_3]$, $g_\beta{}^{(0)}=\tau_\beta$, and $\beta$ is the isospin subscript for the pion. The $O_n$'s are the gauge invariant spin matrices used in Refs. (5) and (6)[7]:

$$O_1=i\gamma_5\gamma\cdot\epsilon\gamma\cdot K, \qquad (2.5a)$$

$$O_2=4i\gamma_5(t-1)^{-1}(P\cdot\epsilon Q\cdot K-P\cdot KQ\cdot\epsilon), \qquad (2.5b)$$

$$O_3=-\gamma_5(\gamma\cdot\epsilon Q\cdot K-\gamma\cdot KQ\cdot\epsilon), \qquad (2.5c)$$

$$O_4=-\gamma_5(\gamma\cdot\epsilon P\cdot K-\gamma\cdot KP\cdot\epsilon), \qquad (2.5d)$$

where $\epsilon$ is the photon polarization vector, $P=\frac{1}{2}(P_1+P_2)$, and $t$ is defined by Eq. (2.6b). The $B_n$'s are those amplitudes which have been shown by Ball to be free of kinematical singularities. They, therefore, admit of a Mandelstam representation and are functions only of the variables $s$, $t$, $u$, which we define as

$$s=-(P_1+K)^2, \qquad (2.6a)$$

$$t=-(Q-K)^2, \qquad (2.6b)$$

$$u=-(P_2-K)^2. \qquad (2.6c)$$

The satisfy the relation $s+u+t=2M^2+1$. The physical interpretation of these variables in the barycentric system of the indicated channel is given below:

$$s=(E_1+k)^2=(E_2+\omega)^2, \qquad (2.7a)$$

$$t=1-2\omega k+2qkz, \qquad (2.7b)$$

$$u=M^2-2E_2k-2qkz, \qquad (2.7c)$$

for channel I, and for channel II,

$$s=M^2-2Ek'-2pk'z', \qquad (2.8a)$$

$$t=(2E)^2=(\omega+k')^2, \qquad (2.8b)$$

$$u=M^2-2Ek'+2pk'z', \qquad (2.8c)$$

where $z=\hat{Q}\cdot\hat{K}$ and $z'=\hat{P}_2\cdot\hat{K}$ are the cosines of the production angles for channel I and II, respectively.

We define the $2\times2$ spin matrices, $\mathfrak{F}$ and $G$, by means of the following scalar products:

$$M\bar{u}(P_2)Tu(P_1)=4\pi(s)^{1/2}\chi^\dagger(N_2)\mathfrak{F}\chi(N_1) \qquad (2.9a)$$

for channel I, and for channel II,

$$M\bar{u}(P_2)Tv(P_1')=4\pi(t)^{1/2}\chi^\dagger(N_2)G\chi(\bar{N}_1), \qquad (2.9b)$$

where the $\chi$'s are Pauli spinors.[8] The differential cross

[5] J. S. Ball, Phys. Rev. **124**, 2014 (1961).

[6] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

[7] The matrices $O_n$ are related to the matrices $M_n$ of Refs. 5 and 6 as follows: $O_1=M_1$, $O_2=2(t-1)^{-1}M_2$, $O_3=-M_3$, and $O_4=-MM_1-\frac{1}{2}M_4$.

[8] We define $G$ such that $\chi(\bar{N})=-\begin{pmatrix}0\\1\end{pmatrix}$ should be used in Eq. (2.9b) when $v(P_1')$ corresponds to an antinucleon spinning up and $\chi(\bar{N})=\begin{pmatrix}1\\0\end{pmatrix}$, when the antinucleon is spinning down.

section for channel I is simply related to $\mathfrak{F}$ as follows:

$$d\sigma/d\Omega=(q/k)\sum|\chi^{\dagger}(N_2)\mathfrak{F}\chi(N_1)|^2, \tag{2.10}$$

and similarly for channel II. We choose to write $\mathfrak{F}$ and $G$ in the forms

$$\mathfrak{F}=\sum_{n=1}^{4} L_n{}^{\mathrm{I}}\mathfrak{F}_n \tag{2.11a}$$

and

$$G=\sum_{n=1}^{4} L_n{}^{\mathrm{II}}G_n, \tag{2.11b}$$

where $L_n{}^{\mathrm{I}}$ and $L_n{}^{\mathrm{II}}$ are defined as in Ref. 5 to be,

$$L_1{}^{\mathrm{I}}=i\boldsymbol{\sigma}\cdot\hat{\epsilon}, \qquad L_2{}^{\mathrm{I}}=\boldsymbol{\sigma}\cdot\hat{Q}\boldsymbol{\sigma}\cdot\hat{K}\times\hat{\epsilon},$$
$$L_3{}^{\mathrm{I}}=i\boldsymbol{\sigma}\cdot\hat{K}\hat{Q}\cdot\hat{\epsilon}, \qquad L_4{}^{\mathrm{I}}=i\boldsymbol{\sigma}\cdot\hat{Q}\hat{Q}\cdot\hat{\epsilon},$$
$$L_1{}^{\mathrm{II}}=\hat{P}_2\cdot\hat{\epsilon}, \qquad L_2{}^{\mathrm{II}}=i\boldsymbol{\sigma}\cdot\hat{P}_2\times\hat{\epsilon},$$
$$L_3{}^{\mathrm{II}}=i\boldsymbol{\sigma}\cdot\hat{P}_2\hat{P}_2\cdot\hat{K}\times\hat{\epsilon}, \qquad L_4{}^{\mathrm{II}}=i\boldsymbol{\sigma}\cdot\hat{K}\times\hat{\epsilon}.$$

Substituting Eqs. (2.4b) and (2.11b) into Eq. (2.9b) and using the linear independence of the $L_n{}^{\mathrm{II}}$, we obtain

$$B_1=[16\pi\sqrt{t}/(t-1)](G_3+G_4), \tag{2.12a}$$

$$B_2=\frac{-8\pi}{(t-4M^2)^{1/2}}\left(G_1+\frac{(t-4M^2)^{1/2}}{2M+\sqrt{t}}G_4+\frac{\sqrt{t}}{(t-4M^2)^{1/2}}G_3\right), \tag{2.12b}$$

$$B_3=\frac{16\pi\sqrt{t}}{(t-1)(t-4M^2)^{1/2}}G_2, \tag{2.12c}$$

$$B_4=\frac{32\pi\sqrt{t}}{(t-1)(2M+\sqrt{t})}\left(G_4+\frac{2M}{2M-\sqrt{t}}G_3\right). \tag{2.12d}$$

The relations between the $\mathfrak{F}_n$'s and $B_n$'s are obtained similarly and have been shown in Ref. 6 to be

$$\mathfrak{F}_1=\frac{s^{1/2}-M}{8\pi\sqrt{s}}[(E_1+M)(E_2+M)]^{1/2}\left[B_1-\tfrac{1}{2}(s^{1/2}+M)B_4+\frac{t-1}{2(s^{1/2}-M)}(B_3-\tfrac{1}{2}B_4)\right], \tag{2.13a}$$

$$\mathfrak{F}_2=\frac{q(s^{1/2}-M)}{8\pi\sqrt{s}}\left(\frac{E_1+M}{E_2+M}\right)^{1/2}\left[-B_1-\tfrac{1}{2}(s^{1/2}-M)B_4+\frac{t-1}{2(s^{1/2}+M)}(B_3-\tfrac{1}{2}B_4)\right], \tag{2.13b}$$

$$\mathfrak{F}_3=\frac{q(s^{1/2}-M)}{8\pi\sqrt{s}}[(E_1+M)(E_2+M)]^{1/2}\times\left[\frac{2(s^{1/2}-M)}{t-1}B_2-B_3+\tfrac{1}{2}B_4\right], \tag{2.13c}$$

$$\mathfrak{F}_4=\frac{q^2(s^{1/2}-M)}{8\pi\sqrt{s}}\left(\frac{E_1+M}{E_2+M}\right)^{1/2}\times\left[\frac{-2(s^{1/2}+M)}{t-1}B_2-B_3+\tfrac{1}{2}B_4\right]. \tag{2.13d}$$

The angular momentum decomposition of these amplitudes is found as in Ref. 6 to be

$$\mathfrak{F}_1=\sum_{J=1/2}\{[(J-\tfrac{1}{2})M_{J-1/2}+E_{J-1/2}]P'_{J+1/2}+[(J+\tfrac{3}{2})M_{J+1/2}+E_{J+1/2}]P'_{J-1/2}\}, \tag{2.14a}$$

$$\mathfrak{F}_2=\sum_{J=1/2}\{(J+\tfrac{1}{2})M_{J-1/2}P'_{J-1/2}+M_{J+1/2}P'_{J+1/2}\}, \tag{2.14b}$$

$$\mathfrak{F}_3=\sum_{J=3/2}\{[E_{J-1/2}-M_{J-1/2}]P''_{J+1/2}+[E_{J+1/2}+M_{J+1/2}]P''_{J-1/2}\}, \tag{2.14c}$$

$$\mathfrak{F}_4=\sum_{J=3/2}\{[M_{J-1/2}-E_{J-1/2}]P''_{J-1/2}-[E_{J+1/2}+M_{J+1/2}]P''_{J+1/2}\}, \tag{2.14d}$$

where $M_{J\pm1/2}$ and $E_{J\pm1/2}$ represent transitions between initial and final states of parity $(-1)^{J\pm1/2}$.

The amplitudes defined by Eq. (2.11b) can also be expanded in terms of amplitudes connecting states of definite parity and angular momentum. Designating these eigenamplitudes by $\alpha_J{}^{\pm}$ and $\beta_J{}^{\pm}$ and using the helicity amplitudes of Jacob and Wick,[9] we obtain

$$G_1=-\sum_J(J+\tfrac{1}{2})\beta_J{}^{-}(t)P'_J, \tag{2.15a}$$

$$G_2=-\tfrac{1}{2}\sum_J\{\alpha_J{}^{-}(t)[JP''_{J+1}+(J+1)P''_{J-1}]-(2J+1)\alpha_J{}^{+}(t)P''_J\}, \tag{2.15b}$$

$$G_3+G_4=-\sum_J(J+\tfrac{1}{2})\beta_J{}^{+}(t)P'_J, \tag{2.15c}$$

$$G_4=-\tfrac{1}{2}\sum_J\{\alpha_J{}^{+}(t)[JP''_{J+1}+(J+1)P''_{J-1}]-(2J+1)\alpha_J{}^{-}(t)P''_J\}, \tag{2.15d}$$

where $P'_J$ means $d/dz'P_J(z')$, etc. Some useful properties of the eigenamplitudes, $\alpha_J{}^{\pm}(t)$ and $\beta_J{}^{\pm}(t)$, are derived in the Appendix. Equations (2.15) are valid for all $z'$ in $E_i$ where $E_i$ is the largest ellipse in the complex $z'$ plane with foci at $\pm1$ in the interior of which $G_i$ is an analytic function of $z'$.[10]

Superscripts (+, −, 0) denoting the isospin character of the quantities in Eqs. (2.11), (2.12), (2.13), (2.14), and (2.15) have been omitted for simplicity.

Chew, Goldberger, Low, and Nambu[6] have obtained crossing relations for the Mandelstam amplitudes.

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[9] M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959).
[10] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1952).

They are

$$B_n^{(i)}(stu)=e_n^{(i)}B_n^{(i)}(uts)\,, \qquad (2.16)$$

where $e_n^{(i)}=+1$ for $i=(0,+)$ and $n=1$, 2, 4 or for $i=(-)$ and $n=3$, and otherwise $e_n^{(i)}=-1$. These amplitudes are related in channel I to the amplitudes leading to final states of total isospin $\frac{1}{2}$ and $\frac{3}{2}$ as follows:

$$B_n^{(+)}=\tfrac{1}{3}B_n^{1/2}+\tfrac{2}{3}B_n^{3/2}\,, \qquad (2.17\text{a})$$

$$B_n^{(-)}=\tfrac{1}{3}B_n^{1/2}-\tfrac{1}{3}B_n^{3/2}\,, \qquad (2.17\text{b})$$

and $B_n^{(0)}$ leads only to states of isospin $\frac{1}{2}$.

## III. COMPLEX ANGULAR MOMENTUM

The steps leading to a Regge description have been discussed extensively for the case of $\pi N$ and $NN$ scattering.[11–13] Since the situation here is essentially the same, we omit certain aspects of these considerations.

Froissart[14] has indicated the appropriate way to continue the eigenamplitudes into the complex $J$ plane. Using the orthogonality properties of the Legendre polynomials and Eqs. (2.15), we find for physical $J$

$$\beta_J^-=(2J+1)^{-1}\int_{-1}^{1}dz(P_{J+1}-P_{J-1})G_1\,, \qquad (3.1\text{a})$$

$$\beta_J^+=(2J+1)^{-1}\int_{-1}^{1}dz(P_{J+1}-P_{J-1})(G_3+G_1)\,, \qquad (3.1\text{b})$$

$$a_J^+\pm a_J^-=-\frac{(2J+1)^{-1}}{J(J+1)}\int_{-1}^{1}(G_4\pm G_2)[(J+1)P_{J-1} +JP_{J+1}\pm(2J+1)P_J]dz\,, \qquad (3.1\text{c})$$

where we have written $P_J$ for $P_J(z)$ and $G_n$ for $G_n(t,z)$. Equations (2.12) are inverted and substituted into Eqs. (3.1). Then we use the $N$-subtracted form of the Mandelstam representation for the $B_n$'s:

$$B_n=\sum_{k=0}^{N-1}\rho_k{}^n(t)z^k+\frac{(-z)^N}{\pi}\int_{z_0}^{\infty}\frac{B_n{}^s(-z't)dz'}{z'^N(z'+z)} +\frac{z}{\pi}\int_{z_0}^{\infty}\frac{B_n{}^u(z't)dz'}{z'^N(z'-z)}\,, \qquad (3.2)$$

where $B_n{}^{s,u}$ is the discontinuity of $B_n$ across the $s$, $u$ cut, $z_0=-E/p$, and $k!\rho_k{}^n(t)=[(\partial^k/\partial z^k)B_n(tz)]_{z=0}$. Interchanging the order of integrations, we obtain for $J\geq N+1$

$$\beta_J^-=\frac{(t-4M^2)^{1/2}}{8\pi^2t(2J+1)}\int_{z_0}^{\infty}dz' \times[b^u(tz')-(-1)^Jb^s(t,-z')] \times[Q_{J+1}-Q_{J-1}]\,, \qquad (3.3\text{a})$$

$$\beta_J^+=\frac{(t-1)}{8\pi^2t(2J+1)}\int_{z_0}^{\infty}dz' \times[B_1{}^u(tz')-(-1)^JB_1{}^s(t,-z')] \times[Q_{J+1}-Q_{J-1}]\,, \qquad (3.3\text{b})$$

$$a_J^+\pm a_J^-=-\frac{(t-1)(2J+1)^{-1}}{8\pi^2tJ(J+1)}\int_{z_0}^{\infty}dz' \times\{\pm[a_\pm{}^u(tz')+(-1)^Ja_\pm{}^s(t,-z')](2J+1)Q_J +[a_\pm{}^u(tz')-(-1)^Ja_\pm{}^s(t,-z')] \times[(J+1)Q_{J-1}+JQ_{J+1}]\}\,, \qquad (3.3\text{c})$$

where

$$a_\pm{}^{u,s}(tz')=2M[B_1{}^{u,s}(tz')-MB_4{}^{u,s}(tz')]+\tfrac{1}{2}tB_4{}^{u,s}(tz') \pm[t(t-4M^2)]^{1/2}B_3{}^{u,s}(tz')\,, \qquad (3.3\text{d})$$

$$b^{u,s}(tz')=(t-1)[B_1{}^{u,s}(tz')-MB_4{}^{u,s}(tz')] +2tB_2{}^{u,s}(tz')\,, \qquad (3.3\text{e})$$

and the $Q_J$'s are Legendre functions of the second kind of argument $z'$.

We now introduce the even and odd eigenamplitudes[11] which will be used in the continuation to complex $J$. We let $\beta_{J,e}{}^+$ represent Eq. (3.3b) for even $J$ and $\beta_{J,0}{}^+$ represent Eq. (3.3b) for odd $J$. This removes the $(-1)^J$ factor. $a_{J,e}{}^\pm$, $a_{J,0}{}^\pm$, $\beta_{J,e}{}^-$, and $\beta_{J,0}{}^-$ are defined similarly. When we wish to indicate explicitly the isospin nature of the eigenamplitudes, we write $\beta_{J,e}{}^{\pm,(i)}$, where $i=(+,-,0)$ and similarly for the others.

We define $\beta_e{}^{\pm,(i)}(Jt)$, $\beta_0{}^{\pm,(i)}(Jt)$,$a_e{}^{\pm,(i)}(Jt)$, and $a_0{}^{\pm,(i)}(Jt)$ to be the continuations of $\beta_{J,e}{}^{\pm,(i)}(t)$, $\beta_{J,0}{}^{\pm,(i)}(t)$, $a_{J,e}{}^{\pm,(i)}(t)$, and $a_{J,0}{}^{\pm,(i)}(t)$, respectively into the complex $J$ plane.[15] Although Eqs. (3.3) have been derived for physical $J$ and $J\geq N+1$, they will represent the continued functions, $\beta_{e,0}{}^\pm(Jt)$ and $a_{e,0}{}^\pm(Jt)$, for all values of $J$ and $t$ for which the integrals converge. Accordingly, $\beta_{e,0}{}^\pm(Jt)$ and $a_{e,0}{}^\pm(Jt)$ will be holomorphic in $J$ for these values of $J$ and $t$, except possibly at the zeros of the denominators of Eqs. (3.3). They are also asymptotically bounded in this domain, as can be seen by using the relation[16]

$$Q_J(z)\underset{J\to\infty}{\longrightarrow}C(z)(J)^{-1/2}[z-(z^2-1)^{1/2}]^J\,, \qquad (3.4)$$

for $z'<1$ and real. What is lacking at this point is a knowledge of the extent of the meromorphy domain for the eigenamplitudes and their properties in this domain. For the scattering of two spinless particles of equal mass, Mandelstam has made considerable progress in this direction, basing his discussion on the existence, for all

[11] V. Singh, Phys. Rev. **129**, 1889 (1963).
[12] S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, Phys. Rev. **126**, 2204 (1962).
[13] Y. Hara, Progr. Theoret. Phys. (Kyoto) **28**, 1048 (1962).
[14] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[15] See K. Bardakci, Phys. Rev. **127**, 1832 (1962) and E. J. Squires, Nuovo Cimento **25**, 242 (1962).
[16] A. Erdelyi *et al.*, in *Higher Transcendental Functions*, (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1.

$J$, of the left-hand discontinuity in $t$ of the partial-wave amplitudes.[17]

It is not the purpose of this paper to improve on the status of such problems. Instead, we assume that the eigenamplitudes have the desired properties (asymptotic boundedness and analyticity, except for Regge poles, to the right of some line $\mathrm{Re}J=-\lambda$, $\lambda>0$) and perform the standard operations necessary to convert Eqs. (2.13) into sums over Regge pole contributions plus background terms (line integrals). For instance, we find that $B_1(st)$ can be written as

$$B_1=16\pi t^{1/2}(t-1)^{-1}(-\tfrac{1}{2})\sum_J(2J+1)\beta_J{}^+P'_J(z'), \quad (3.5a)$$

$$B_1=16\pi\tfrac{1}{2}t^{1/2}(1-t)^{-1}\sum_{\sigma,J}(2J+1)\beta_{J,\sigma}{}^+(\tfrac{1}{2})\times[P'_J(z')-\sigma P'_J(-z')], \quad (3.5b)$$

where $\sigma$ is the signature and $\frac{1}{2}[P'_J(z')-\sigma P'_J(-z')]$ is a projection operator for even $J$ if $\sigma$ is even ($\sigma=+1$) and for odd $J$ if $\sigma$ is odd. This sum can be transformed into a contour integral in the complex $J$ plane using Cauchy's theorem:

$$B_1=16\pi\frac{\sqrt{t}}{2(1-t)}\sum_\sigma\frac{\sigma}{2i}\oint_C\frac{dJ(2J+1)}{\sin\pi J}\times\beta_\sigma{}^+(J)\tfrac{1}{2}[P'_J(z')-\sigma P'_J(-z')], \quad (3.6)$$

where $C$ is a path encircling the real axis from $J=1$ to $J=\infty$. Deforming the path, we obtain

$$B_1=16\pi\frac{\sqrt{t}}{2(1-t)}\sum_{\sigma,\alpha(\sigma)}\pi\sigma R_{\alpha(\sigma)}[2\alpha(\sigma)+1]\times\frac{[P'_{\alpha(\sigma)}(z')-\sigma P'_{\alpha(\sigma)}(-z')]}{2\sin\pi\alpha(\sigma)}+\begin{pmatrix}\text{background}\\\text{term}\end{pmatrix}, \quad (3.7)$$

where $\alpha(\sigma)$ and $R_{\alpha(\sigma)}$ are the poles and residues, respectively, of $\beta_\sigma{}^+(Jt)$. Similar expressions may be obtained for the other Mandelstam amplitudes. The next step is to let $z'$ and $s$ approach infinity for fixed $t$. It is important to realize that this could not have been done in Eq. (3.5) or (3.6). These equations are valid only for $z'$ within one of the ellipses denoted by $E_i$. In fact, the contribution along the infinite semicircle which has been dropped in going from Eq. (3.6) to Eq. (3.7) does not vanish for large $z'$ as it does for $z'<1$. Equation (3.7), however, is valid for arbitrarily large $z'$ if the Regge trajectories are bounded. This follows from recognizing that the background term is the product of a $z^{-\lambda}$ factor and a Fourier integral of the form

$$\int_{-\infty}^{\infty}d(\mathrm{Im}J)f(-\lambda+i\,\mathrm{Im}J,t)\exp[i\,\mathrm{Im}J\ln z], \quad (3.8)$$

[17] S. Mandelstam, Ann. Phys. (N. Y.) **21**, 302 (1963).

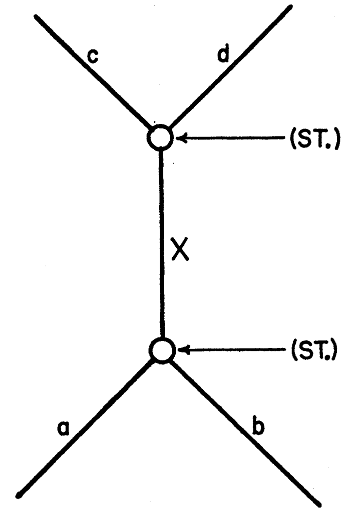


FIG. 1. Trajectories for the strongly interacting process, $a+b\to c+d$.

which vanishes as $z\to\infty$ by the Riemann-Lebesgue theorem if a path, $\mathrm{Re}J=-\lambda$, is chosen such that the integrand is bounded along it.

We mention in passing, that Eq. (3.6) is really not correct for $t\leq 4$ since the Regge poles migrate to the real axis for $t$ below threshold. The contour $C$ may therefore enclose some Regge poles as well as the poles of $CSC(\pi J)$. However, these are easily included, and the form of Eq. (3.7) is not altered as long as all the Regge poles with $\mathrm{Re}\alpha>-\lambda$ are included in the summation.

## IV. REGGE TRAJECTORIES IN CHANNEL II

In nonrelativistic Schrödinger theory, the angular momentum poles of an eigenamplitude have a simple physical interpretation; if $\mathrm{Re}\alpha(E)$ passes through zero or a positive integer at an energy $E=E'$, then there exists a solution to Schrödinger's equation of energy $E'$ which describes a bound state if $E'<0$ and a shadow state[18] if $E'>0$. To make this interpretation applicable to relativistic strong interaction physics, the concept of a bound state or resonance is defined in the conventional manner. Figure 1 illustrates this definition; $X$ can be any composite particle or resonance whose quantum numbers are such that $B$, $I$, $P$, $G$, $C$,[19] $S$, and charge are conserved at each vertex. This definition is readily extended to include reactions in which more than one type of interaction participates. For example, both strong and electromagnetic interactions are present in the photoproduction of pions and Fig. 2 illustrates the definition used in this case; at the $Xbc$ vertex, $B$, $I$, $P$, $G$, $C$, $S$, and charge are conserved as before, but at the

[18] T. Regge, (see Ref. 2). Although the shadow state, real $E$ and complex $J$, is not identical to the resonant states of Breit and Wigner, real $J$ and complex $E$, they may be identified for our purposes since they both induce a resonance type behavior into the physical eigenamplitudes.

[19] The eigenvalue $C$ of the charge conjugation operator is an appropriate quantum number only for neutral states.

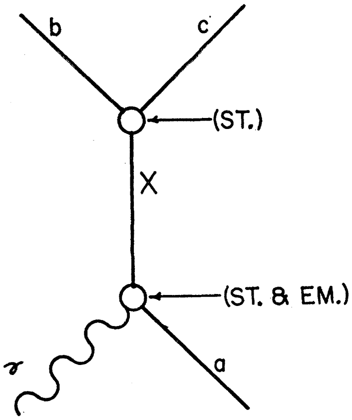


FIG. 2. Trajectories for the process, $\gamma+a\rightarrow b+c$, involving both strong and electromagnetic interactions.

$abX$ vertex only $B$, $P$, $C$, $S$ and charge need be conserved. Again, $X$ must be a composite particle or a resonance. Observe that if we were doing conventional field theory, $X$ would not be restricted to only these multiparticle states which resonate.

For notational convenience we label the four charge configurations of channel II

$$\gamma+\pi^0\rightarrow p+\bar{p}\,,$$
$$\gamma+\pi^0\rightarrow n+\bar{n}\,,$$
$$\gamma+\pi^-\rightarrow n+\bar{p}\,,$$

and

$$\gamma+\pi^+\rightarrow p+\bar{n}$$

as II(1), II(2), II(3), and II(4), respectively. Four of the eigenamplitudes defined in Sec. II represent transitions between states of negative parity and the other four between states of positive parity. In the Appendix it is shown that the latter are $a_e^+$, $\beta_e^+$, $a_0^-$, and $\beta_0^-$. These can be eliminated from our consideration for the following reasons. The ABC,[20] $K_1K_1$,[21] and the two vacuum trajectories are the only known $B=S=0$ trajectories of even parity. They are not permitted in reactions II(1) and II(2) because of conservation of $C$ and in reactions II(3) and II(4) because of conservation of charge.

The four remaining eigenamplitudes corresponding to negative parity transitions are $a_0^+$, $\beta_0^+$, $a_e^-$, and $\beta_e^-$. As before, we must have $B=S=0$. At present there are only five known odd-parity trajectories with $B=S=0$, the $\rho$, $\phi$,[22] $\omega$, $\pi$ and $\eta$ trajectories. The latter is excluded for the same reasons as were the even parity trajectories. Since the $\rho$, $\phi$, and $\omega$ trajectories have odd signature (spin 1), the only trajectory available to $a_e^-$ and $\beta_e^-$ is the $\pi$ trajectory (even signature); and it is not found in $a_e^-$ because in $I=1$ transitions $a_e^-$ leads to final states with even $G$ parity (see Appendix). Although $G$ parity is not conserved in general, it is conserved at the $XN\bar{N}$ vertex, as shown in Fig. 2. Furthermore, the pion trajectory is not present in II(1) and II(2) because of conservation of $C$. In fact, $\beta_e^-$ is identically zero in these reactions for the same reason. This means that $\beta_e^{-,(+)}$ and $\beta_e^{-,(0)}$ must vanish [see Eqs. (5.11)]. The two odd amplitudes $a_0^{+,(i)}$ and $\beta_0^{+,(i)}$ share the same Regge trajectories[23] but their residues may differ. For reactions II(1) and II(2), the $\rho$, $\phi$, and $\omega$ trajectories are allowed. However, reactions II(3) and II(4) involve pure $I=1$ final states so that the $\omega$ and $\phi$ trajectories ($I=0$) are forbidden.

If we let $n=1$ in Eq. (2.16) and use Eq. (2.12a), we obtain

$$G_3^{(-)}(stu)+G_4^{(-)}(stu)=-[G_3^{(-)}(uts)+G_4^{(-)}(uts)].$$

Since interchanging $s$ and $u$ in channel II is equivalent to replacing $z'$ by $-z'$ and since $P_J'(-z')=(-1)^{J+1}P_J'(z')$, it is evident that $a_0^{+,(-)}$ and $\beta_0^{+,(-)}$ must vanish. We conclude then that the $\rho$ trajectory occurs in $a_0^{+,(0)}$ and $\beta_0^{+,(0)}$, whereas the $\omega$ and $\phi$ trajectories are found in $a_0^{+,(+)}$ and $\beta_0^{+,(+)}$. The $\rho$ trajectory is not allowed in $a_0^{+,(+)}$ or $\beta_0^{+,(+)}$ because these transitions lead only to final states with zero isospin.[5]

## V. ASYMPTOTIC SYMMETRIES

### A. Forward Scattering

In Part A of this section, we investigate the asymptotic behavior of the channel I scattering amplitudes in the forward direction. For unpolarized nucleons and photons the relation between these amplitudes and the differential cross section is

$$(d\sigma/d\Omega)=(q/k)\{|\mathfrak{F}_1|^2+|\mathfrak{F}_2|^2-2z\,\mathrm{Re}\mathfrak{F}_1^*\mathfrak{F}_2+(1-z^2)[\tfrac{1}{2}|\mathfrak{F}_3|^2+\tfrac{1}{2}|\mathfrak{F}_4|^2+\mathrm{Re}(\mathfrak{F}_1^*\mathfrak{F}_4+\mathfrak{F}_2^*\mathfrak{F}_3+z\mathfrak{F}_3^*\mathfrak{F}_4)]\}\,. \quad (5.1)$$

If we fix $z$ at one, this reduces to $(d\sigma/d\Omega)=(q/k)|A_1|^2$, where $A_1=\mathfrak{F}_1+\mathfrak{F}_2$, but then

$$z'=\frac{2(s-M^2)t^{1/2}+(t-1)\sqrt{t}}{-(t-1)(t-4M^2)^{1/2}}\xrightarrow[s\to\infty]{}1-\frac{(\mathrm{const})}{s}\,, \quad (5.2)$$

and the approximation, $P_\alpha'(z')\rightarrow z'^{\alpha-1}$, which leads to the usual Regge equations, cannot be used. Therefore, we consider fixed $t$ in which case $z'\xrightarrow[s\to\infty]{}(1/M)s(-t)^{1/2}$ as desired, but we now are approaching the forward direction of channel I only asymptotically:

$$z=\frac{(t-1)\sqrt{s}}{q(s-M^2)}+\frac{s-M^2+1}{2q\sqrt{s}}\xrightarrow[s\to\infty]{}1+\frac{2t}{s}\,. \quad (5.3)$$

[20] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters **5**, 258 (1960); **7**, 35 (1961) and Rev. Mod. Phys. **33**, 393 (1961).

[21] G. Alexander, O. Dahl, L. Jacobs, G. Kalbfleisch, D. Miller, *et al.*, Phys. Rev. Letters **9**, 460 (1962).

[22] L. Bertanza, V. Brisson, P. L. Connolly, E. L. Hart, I. S. Mittra *et al.*, Phys. Rev. Letters **9**, 180 (1962).

[23] J. M. Charap and E. J. Squires, Phys. Rev. **127**, 1387 (1962).

Therefore, all the terms in Eq. (5.1) must be included (not just $A_1$).

For this purpose it is convenient to rewrite Eq. (5.1) in the form

$$s(d\sigma/dt) \underset{s\to\infty}{\longrightarrow} -4\pi\{|A_1|^2 - 8t^2(s)^{-2}\,\mathrm{Re}\,\mathfrak{F}_3{}^*\mathfrak{F}_4 - 2t(s)^{-1}[2\,\mathrm{Re}(\mathfrak{F}_1{}^*\mathfrak{F}_2 + \mathfrak{F}_1{}^*\mathfrak{F}_4 + \mathfrak{F}_2{}^*\mathfrak{F}_3) + |\mathfrak{F}_3 + \mathfrak{F}_4|^2]\}, \quad (5.4)$$

where Eq. (5.3) and $(d\sigma/dt) \underset{s\to\infty}{\longrightarrow} -4\pi(s)^{-1}(d\sigma/d\Omega)$ have been used. We have seen in Sec. III that the Regge contributions arising in channel II may be continued by way of the Mandelstam amplitudes into the physical region of channel I. This enables us to express each term of Eq. (5.4) as a function of the $\rho$, $\phi$, $\omega$, and $\pi$ Regge poles since the $\mathfrak{F}_n$'s are given in terms of the $B_n$'s by Eqs. (2.13).

Consider the first term of Eq. (5.4), for instance. Although it is related to the $B_n$'s in a complicated (algebraically) manner, considerable simplification results if we confine our attention to large $s$. Using the expansion $[(E_1+M)(E_2+M)]^{1/2} \underset{s\to\infty}{\longrightarrow} q \underset{s\to\infty}{\longrightarrow} \frac{1}{2}\sqrt{s}$ and Eqs. (2.13a) and (2.13b), we have for fixed $t$,

$$A_1{}^{(i)} \underset{s\to\infty}{\longrightarrow} \tfrac{1}{16}(\pi)^{-1}s^{1/2}(2B_1{}^{(i)} - MB_4{}^{(i)}), \quad (5.5)$$

where only the largest term has been retained. The relative magnitude of the $B_n$'s is established by consulting Eqs. (2.12) and (2.15). For example, $B_3$ is proportional to $G_2$ which behaves as $s^{\alpha-2}$ for large $s$ since we do not know of any Regge trajectories which belong to $a^-(Jt)$. Similarly, it is seen that the other $B_n$'s are of the order $s^{\alpha-1}$ for large $s$. We temporarily ignore the $B_4$ term of Eq. (5.5) in order to simplify the presentation to follow. If we let $s$ tend to infinity for fixed $t$ in Eq. (3.7),

$$2B_1{}^{(i)}(st) \underset{s\to\infty}{\longrightarrow} 16\pi \sum_{\alpha_0} \eta^{(i)}(\alpha_0)\xi(\alpha_0)s^{\alpha_0(t)-1}, \quad (5.6a)$$

and substitute this into Eq. (5.5), we obtain

$$A_1{}^{(i)}(st) \underset{s\to\infty}{\longrightarrow} \sum_{\alpha_0} \eta^{(i)}(\alpha_0)\xi(\alpha_0)s^{\alpha_0(t)-1/2}, \quad (5.6b)$$

where $\xi(\alpha)$ is the signature factor $2\xi(\alpha)\sin\pi\alpha = 1-e^{-i\pi\alpha}$, $\eta(\alpha)$ is the residue function

$$\pi^{1/2}(2pk')^{\alpha-1}\eta^{(i)}(\alpha) = -t^{1/2}(t-1)^{-1}\alpha R_\alpha{}^{(i)}(2\alpha+1)2^\alpha \times \sin(\pi\alpha)\Gamma(-\alpha)\Gamma(\alpha+\tfrac{1}{2}), \quad (5.6c)$$

$R_\alpha{}^{(i)}(t)$ is the residue of $\beta^{+,(i)}(Jt)$ at $J=\alpha(t)$, and $\Gamma(x)$ is a gamma function. We have omitted the sum over $\alpha_e$ because no physical manifestations of these trajectories have been observed. Furthermore, the sum over $\alpha_0$ reduces to at most two terms for each value of (i) if only the trajectories corresponding to the known "elementary" particles and resonances are included:

$$A_1{}^{(+)} \underset{s\to\infty}{\longrightarrow} \eta_\phi(t)\xi_\phi s^{\alpha_\phi(t)-1/2} + \eta_\omega(t)\xi_\omega s^{\alpha_\omega(t)-1/2}, \quad (5.7a)$$

$$A_1{}^{(0)} \underset{s\to\infty}{\longrightarrow} \eta_\rho(t)\xi_\rho s^{\alpha_\rho(t)-1/2}, \quad (5.7b)$$

and $A_1{}^{(-)}$ vanishes, i.e., there are no trajectories to contribute to $A_1{}^{(-)}$; the pion trajectory contributes only to $\mathfrak{F}_3$ and $\mathfrak{F}_4$. Equations (5.7) are changed by the inclusion of the $B_4$ term only to the extent that the $\eta$'s will be different. In particular, $R_\alpha$ in Eq. (5.6c) will be replaced by $(4M^2-t)^{-1}[(2M^2-t)R_\alpha + M(t)^{1/2}\alpha R_\alpha']$, where $R_\alpha'$ is the residue of $a^+$ at $J=\alpha(t)$. If the $t$ dependence of $R_\alpha$ and $R_\alpha'$ is the same near $t=0$, then this replacement is simply $\frac{1}{2}R_\alpha$ for small $t$.

We now consider the high-energy contributions coming from the other terms of Eq. (5.4). We begin by using Eq. (2.13) to obtain

$$\mathrm{Re}(\mathfrak{F}_1{}^*\mathfrak{F}_2 + \mathfrak{F}_1{}^*\mathfrak{F}_4 + \mathfrak{F}_2{}^*\mathfrak{F}_3) \underset{s\to\infty}{\longrightarrow} 2s\,\mathrm{Re}A_3{}^*A_2, \quad (5.8a)$$

$$|\mathfrak{F}_3 + \mathfrak{F}_4|^2 \underset{s\to\infty}{\longrightarrow} s|A_4|^2, \quad (5.8b)$$

$$\mathrm{Re}(\mathfrak{F}_3{}^*\mathfrak{F}_4) \underset{s\to\infty}{\longrightarrow} -|sA_2|^2, \quad (5.8c)$$

where

$$16\pi(t-1)A_2 = (s)^{1/2}B_2,$$

$$A_4 = (2M)^{-1}(A_1 + 2A_3 - 4M^2A_2),$$

and

$$16\pi A_3 = (s)^{1/2}(-B_1 + MB_4).$$

The latter is not essentially different from $A_1$, and, in fact, Eqs. (5.7) may be used for $A_3$ if new residue functions are substituted in place of the $\eta$'s. These new functions, $\eta_\alpha'$, should defined by Eq. (5.6c) with $R_\alpha$ replaced by $\frac{1}{2}(4M^2-t)^{-1}[2M(t)^{1/2}\alpha R_\alpha' - tR_\alpha]$. The remaining amplitude $A_2$ may be conveniently expressed as

$$2A_2 = -s^{1/2}(t-1)^{-1}(t-4M^2)^{-1/2}G_1 + t^{-1}A_3, \quad (5.9)$$

where Eq. (2.12b) has been used. Since $\beta_J{}^{-,(+,0)}=0$ and $A_3{}^{(-)}$ vanishes for large $s$, Eq. (5.9) implies that $2tA_2{}^{(+,0)} = A_3{}^{(+,0)}$ and

$$A_2{}^{(-)} \underset{s\to\infty}{\longrightarrow} \eta_\pi(t)\xi_\pi s^{\alpha_\pi(t)-1/2}, \quad (5.10)$$

where $\eta_\pi$ is given by Eq. (5.6c) with $R_\alpha$ replaced by $\frac{1}{4}R_\pi(t)[t(t-4M^2)]^{1/2}$.

We define the isospin amplitudes $A_n{}^1$, $A_n{}^2$, $A_n{}^3$, and $A_n{}^4$ to represent the reactions $\gamma+p\rightarrow\pi^0+p$, $\gamma+n\rightarrow\pi^0+n$, $\gamma+p\rightarrow\pi^++n$, and $\gamma+n\rightarrow\pi^-+p$, respectively. We designate these in general by $A_n{}^k$. It has been shown in Ref. (6) that these amplitudes are related to the $A_n{}^{(i)}$ amplitudes as follows:

$$A_n{}^{1,2} = A_n{}^{(+)} \pm A_n{}^{(0)}, \quad (5.11a)$$

$$A_n{}^{3,4} = \sqrt{2}(A_n{}^{(0)} \pm A_n{}^{(-)}), \quad (5.11b)$$

where the upper sign in Eq. (5.11a) is for $A_n{}^1$ and in Eq. (5.11b), for $A_n{}^3$, etc. In this notation the differential cross section is given asymptotically by

$$s(d\sigma/dt)_k \xrightarrow[s\to\infty]{} -4\pi\{|A_1^k|^2-2t|A_4^k|^2 -8t^2|A_2^k|^2-4t\,\mathrm{Re}[A_3^k(A_2^k)^*]\}. \quad (5.12)$$

For $k=1$ this describes $\gamma+p\to\pi^0+p$, etc. From Eqs. (5.7), (5.10), and (5.11) we see that $(d\sigma/dt)_k$ depends only on the $\rho$ and $\pi$ trajectories for $k=3$, 4, whereas for $k=1$, 2 it depends on the $\rho$, $\phi$, and $\omega$ trajectories. Its explicit dependence on these trajectories is obtained by substituting Eq. (5.7), Eq. (5.10), and the corresponding equations for $A_3$ and $A_2^{(+,0)}$ into Eq. (5.12):

$$\left(\frac{d\sigma}{dt}\right)_{1,2} \xrightarrow[s\to\infty]{} F_{\phi\omega}(t)\left(\frac{s}{s_0}\right)^{\alpha_\phi+\alpha_\omega-2} \pm F_{\rho\phi}(t)\left(\frac{s}{s_0}\right)^{\alpha_\rho+\alpha_\phi-2} \pm F_{\rho\omega}(t)\left(\frac{s}{s_0}\right)^{\alpha_\rho+\alpha_\phi-2} + \sum_{r=\rho,\phi,\omega} F_r(t)\left(\frac{s}{s_0}\right)^{2(\alpha_r-1)}, \quad (5.13a)$$

$$\left(\frac{d\sigma}{dt}\right)_{3,4} \xrightarrow[s\to\infty]{} 2\left[F_\rho(t)\left(\frac{s}{s_0}\right)^{2(\alpha_\rho-1)} + F_\pi(t)\left(\frac{s}{s_0}\right)^{2(\alpha_\pi-1)} \pm F_{\rho\pi}(t)\left(\frac{s}{s_0}\right)^{\alpha_\rho+\alpha_\pi-2}\right], \quad (5.13b)$$

where the upper sign is for $(d\sigma/dt)_1$ in Eq. (5.13a) and for $(d\sigma/dt)_3$ in Eq. (5.13b), etc. The $F(t)$'s are defined by the above equations and are functions of $\alpha$, $\eta_\alpha$, $\eta_\alpha'$, $\xi_\alpha$, and $t$; for example, $F_\pi(t)=32\pi t(t+M^2)|\eta_\pi\xi_\pi|^2 s_0^{2(\alpha_\pi-1)}$. They are a measure of the strength of the "coupling of the trajectories" at the two vertexes of Fig. 1 and may be determined empirically.

### B. Backward Scattering

The high-energy scattering in the backward direction is controlled by the Regge trajectories of the same channel (or alternatively by those in channel III). We do not present a separate discussion of these trajectories as was done in Sec. IV for the trajectories of channel II since the situation is much simpler in this case. The trajectories in channel I are labeled by the quantum numbers, $B=1$, $S=0$, $I=\frac{1}{2}$, $\frac{3}{2}$, and $P=\pm1$. At present there are only three such trajectories. If we characterize them by their first member, they may be called the nucleon trajectory, the 3, 3 resonance trajectory, and the trajectory for the 600-MeV $\pi p$ resonance. They are discussed in detail by Singh *et al.*[11]

To introduce these trajectories into our description of backward scattering, it is necessary to carry out the same kind of manipulations discussed in part A of this section. We assume that the even and odd eigenamplitudes for channel I [introduced in Eqs. (2.14)] have the appropriate properties for complex $J$, write Eqs. (2.14) as contour integrals, and deform the path to allow the Regge pole contributions to enter. If we let $u$ tend to infinity and keep only the largest term, we find that $\mathfrak{F}_1$ and $u\mathfrak{F}_3$ behave as $u^{\alpha_N(s)-1/2}$, while $\mathfrak{F}_2$ and $u\mathfrak{F}_4$ behave as $u^{\alpha_{33}(s)-1/2}$, where $\alpha_N$ and $\alpha_{33}$ are the nucleon and 3,3 resonance trajectories, respectively. When Eqs. (2.13) are inverted it is found that $\mathfrak{F}_3$ and $\mathfrak{F}_4$ are always multiplied by a factor of $u$ compared to $\mathfrak{F}_1$ and $\mathfrak{F}_2$. In particular, we obtain the following results for fixed $s$:

$$B_n^{(+)}(su) \xrightarrow[u\to\infty]{} N_n^{(+)}(s)\xi_N u^{\alpha_N(s)-1/2} + T_n^{(+)}(s)\xi_{33}u^{\alpha_{33}(s)-1/2}, \quad (5.14a)$$

$$B_n^{(0)}(su) \xrightarrow[u\to\infty]{} N_n^{(0)}(s)\xi_N u^{\alpha_N(s)-1/2}, \quad (5.14b)$$

$$B_n^{(-)}(su) \xrightarrow[u\to\infty]{} N_n^{(-)}(s)\xi_N u^{\alpha_N(s)-1/2} + T_n^{(-)}(s)\xi_{33}u^{\alpha_{33}(s)-1/2}, \quad (5.14c)$$

where $N_n^{(i)}(s)$ and $T_n^{(i)}(s)$ are the residue functions for the nucleon and the 3,3 trajectories, respectively. Since the 3,3 resonance has $I=\frac{3}{2}$, the $B_n^{(0)}$ amplitude does not contain a contribution from the 3,3 trajectory [see Eqs. (2.17)]. We now interchange $s$ and $u$ in Eqs. (5.14) and use the crossing relations, Eq. (2.16), to obtain expressions for the Mandelstam amplitudes $B_n^{(i)}(su)$ evaluated in the physical region of channel I, e.g.,

$$B_n^{(0)}(su) \xrightarrow[s\to\infty]{} e_n^{(0)}N_n^{(0)}(u)\xi_N s^{\alpha_N(u)-1/2}. \quad (5.15)$$

The combinations of these isospin amplitudes appropriate for the description of the four charge states of channel I are given by Eqs. (5.11), and we define the amplitudes $B_n^k$ for $k=1$, 2, 3, 4 in accordance with the definitions of $A_n^k$. Expressing the $B_n^k(su)$'s in terms of the nucleon and 3,3 trajectories, we have

$$B_n^{1,2}(su) \xrightarrow[s\to\infty]{} e_n^{(0)}\{[N_n^{(+)}\pm N_n^{(0)}]\xi_N s^{\alpha_N-1/2} + T_n^{(+)}\xi_{33}s^{\alpha_{33}-1/2}\} \quad (5.16a)$$

and

$$B_n^{3,4}(su) \xrightarrow[s\to\infty]{} \sqrt{2}\{[e_n^{(0)}N_n^{(0)}\pm e_n^{(-)}N_n^{(-)}]\xi_N s^{\alpha_N-1/2} \pm e_n^{(-)}T_n^{(-)}\xi_{33}s^{\alpha_{33}-1/2}\}, \quad (5.16b)$$

where the upper sign is for $B_n^1$ in Eq. (5.16a) and for $B_n^3$ in Eq. (5.16b). For large $s$ and fixed $u$, $z$ behaves as $-1-2u/s$, $B_3^{(i)}(su)$ tends to $-\frac{1}{2}B_4^{(i)}(su)$, and Eq. (5.1) becomes

$$\left(\frac{d\sigma}{du}\right)_k \xrightarrow[s\to\infty]{} -\frac{1}{8\pi s}\mathrm{Re}B_1^k(B_2^k)^* + \sum_{n=1}^{4}\frac{C_n}{s}|B_n^k|^2, \quad (5.17)$$

where $16\pi C_1=u$, $8\pi C_2=-u$, $C_3=0$, and $64\pi C_4=M^4+u^2$. Substituting Eqs. (5.16) into Eq. (5.17), we find that the differential cross sections describing backward scattering in the four charge configurations are given by

$$\left(\frac{d\sigma}{du}\right)_k \xrightarrow[s\to\infty]{} F^k{}_{33}(u)\left(\frac{s}{s_0}\right)^{2[\alpha_{33}-1]} + F^k{}_{33\,N}(u)\left(\frac{s}{s_0}\right)^{\alpha_N+\alpha_{33}-2} + F^k{}_N(u)\left(\frac{s}{s_0}\right)^{2[\alpha_N-1]}, \quad (5.18)$$

where $F_{33}{}^1(u)=F_{33}{}^2(u)$ and $F_{33}{}^3(u)=F_{33}{}^4(u)$.

### C. Discussion

On the basis of the preceding remarks we are now able to make some observations regarding the high-energy symmetries exhibited by the scattering amplitudes for the various charge states of channel I. If the dominance of the $\rho$ over the $\pi$ trajectory is assumed for small $t$, it is clear that for the forward direction the asymptotic equality of the differential cross sections for the photoproduction of pions of positive and negative charge is predicted. More specifically,

$$\left(\frac{d\sigma}{dt}\right)_3 \xrightarrow[s\to\infty]{} \left(\frac{d\sigma}{dt}\right)_4 \xrightarrow[s\to\infty]{} 2F_\rho(t)\left(\frac{s}{s_0}\right)^{2[\alpha_\rho(t)-1]}. \quad (5.19)$$

The value of $s$ at which this effect should be noticed depends, among other things, on the relative magnitude of $F_\rho(t)$ and $F_{\pi\rho}(t)$, and there is some indication that the coupling of the $\rho$ to $N\bar{N}$ states is small[24] (compared to that of the $\omega$ for instance). For the photoproduction of neutral pions, $\gamma+p\rightarrow\pi^0+p$ and $\gamma+n\rightarrow\pi^0+n$, we have two competing trajectories of comparable magnitude, the $\rho$ and $\omega$, and possibly a third, the $\phi$, and we expect this to delay [relative to the advent of Eq. (5.19)] the appearance of a corresponding asymptotic symmetry for this case unless $F_{\pi\rho}(t)$ is considerably larger than $F_{\rho\omega}(t)$ [and perhaps $F_{\phi\rho}(t)$]. However, if one of these two trajectories is even slightly larger than the other, it is necessary that the differential cross sections for these reactions eventually display a symmetry of this type.

For scattering in the backward direction, the existence of symmetries such as this depends on the dominance of the 3,3 trajectory over the nucleon trajectory. If this is assumed, the following asymptotic equalities are obtained for small $u$:

$$\left(\frac{d\sigma}{du}\right)_1 \xrightarrow[s\to\infty]{} \left(\frac{d\sigma}{du}\right)_2 \xrightarrow[s\to\infty]{} F_{33}{}^1(u)\left(\frac{s}{s_0}\right)^{2[\alpha_{33}(u)-1]} \quad (5.20a)$$

and

$$\left(\frac{d\sigma}{du}\right)_3 \xrightarrow[s\to\infty]{} \left(\frac{d\sigma}{du}\right)_4 \xrightarrow[s\to\infty]{} F_{33}{}^3(u)\left(\frac{s}{s_0}\right)^{2[\alpha_{33}(u)-1]}. \quad (5.20b)$$

Although the backward scattering in all four charge states is controlled by the same Regge trajectory, the asymptotic equality among the differential cross sections (for backward scattering) is incomplete in that $F_{33}{}^1(u)\neq F_{33}{}^3(u)$. Contrasting this with the situation which prevails for forward scattering, we see that the dominance of the $\rho$ trajectory in all four charge states would imply the asymptotic equality in the forward direction of all four differential cross sections (apart from a factor of 2).

Equations (5.19) and (5.20) may be regarded as the generalization of the Pomeranchuk relations to the photoproduction of pions (in the sense mentioned in the Introduction). However, this is not the canonical generalization, which relates to the amplitudes for the direct and crossed channels. Instead, the individual charge states are related, and an asymptotic version of charge independence for forward and backward scattering is obtained (except possibly for the forward production of neutral pions).

In conclusion, we remark that the results obtained in this paper pertaining to the photoproduction of neutral pions in the forward direction need no essential modification for application to the photoproduction of $\eta$ mesons. There is considerable difference however between the photoproduction of $\pi^0$ and $\eta$ mesons in the backward direction. This can be understood as follows: With the exception of isospin and $G$ parity, the $\pi$ and $\eta$ mesons are described by the same quantum numbers. The same Regge trajectories will contribute to the forward production of $\eta$'s as to the forward production of $\pi^0$'s since these trajectories come from channel II, $\gamma+\eta\rightarrow N+\bar{N}$ or $\gamma+\pi^0\rightarrow N+\bar{N}$, and $I$ and $G$ are not conserved at the vertices involving the $\eta$ and $\pi^0$, $\gamma\eta X$ and $\gamma\pi^0X$. On the other hand, the trajectories which contribute to backward scattering come from the channel, $\gamma+N\rightarrow\pi^0+N$ or $\gamma+N\rightarrow\eta+N$, and $I$ and $G$ are now conserved at the vertexes containing the $\eta$ and $\pi^0$. In particular, this eliminates the isospin-$\frac{3}{2}$ trajectories (which are allowed in backward $\pi^0$ production) from the description of backward $\eta$ production. This means that the differential cross sections for the two processes, $\gamma+N\rightarrow\pi^0+N$ and $\gamma+N\rightarrow\eta+N$ will display the same energy dependence in the forward direction (for large $s$ and fixed $t$), whereas the differential cross section for $\pi^0$ production will have a stronger energy dependence in the backward direction than the one for backward production of $\eta$'s (for large $s$ and fixed $u$).


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#### APPENDIX

The eigenamplitudes $a_J{}^{\pm}$ and $\beta_J{}^{\pm}$ represent transitions between initial and final states of angular momentum $J$. The final states are also eigenstates of $G$, $P$,

[24] S. D. Drell, in *Proceedings of the 1962 International Conference on High-Energy Physics, CERN* (CERN, Geneva, 1962), p. 897.

and $C$ (the latter only for states of zero charge). Their eigenvalues are obtained below.

The eigenamplitudes are defined in Ref. 5 to be

$$(2pk')^{1/2}J(J+1)a_J{}^{\pm}=T_J(+,-,1)\pm T_J(-,+,1),$$
$$(2pk')^{1/2}[J(J+1)]^{1/2}\beta_J{}^{\pm}=T_J(+,+,1)\pm T_J(-,-,1),$$

where

$$T_J[\lambda(N),\lambda(\bar{N}),\lambda(\gamma)]=\langle\lambda(N),\lambda(\bar{N}),J\,||\,T\,||\,J,\lambda(\gamma),\lambda(\pi)\rangle,$$

$\lambda(A)$ is the helicity of particle $A$, and we write $T_J(+,+,1)$ rather than $T_J(\frac{1}{2},\frac{1}{2},1)$, etc. We now express the nucleon, antinucleon states of definite helicity $|JM\lambda(N)\lambda(\bar{N})\rangle$ in terms of the states $|JMLS\rangle$, where $L$ is the relative orbital momenta and $S$ is the total spin. Using Eq. (B5) of Ref. 9, we find that

$$\sqrt{2}\,|J,M,\pm,\pm\rangle$$
$$=\pm|J,M,J,0\rangle+\left(\frac{J}{2J+1}\right)^{1/2}|J,M,J-1,1\rangle-\left(\frac{J+1}{2J+1}\right)^{1/2}|J,M,J+1,1\rangle,$$
$$\sqrt{2}\,|J,M,\pm,\mp\rangle$$
$$=\mp|J,M,J,1\rangle+\left(\frac{J-1}{2J+1}\right)^{1/2}|J,M,J-1,1\rangle+\left(\frac{J+2}{2J+1}\right)^{1/2}|J,M,J+1,1\rangle.$$

TABLE I. Odd-parity transition amplitudes.

| | $a_0{}^+,\beta_0{}^+$ | | | $a_e{}^-$ | | | $\beta_e{}^-$ | | |
|---|---|---|---|---|---|---|---|---|---|
| | II$(k)$[a] | | II$(j)$[b] | II$(k)$[a] | | II$(j)$[b] | II$(k)$[a] | | II$(j)$[b] |
| $I$ | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| $G$ | −1 | 1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 |
| $C$ | −1 | −1 | $X$ | −1 | −1 | $X$ | 1 | 1 | $X$ |
| Regge traj. | $\omega,\phi$ | $\rho$ | $\rho$ | none | | none | none | | $\pi$ |

[a] The index $k$ may be 1 or 2 in the symbol II$(k)$.
[b] The index $j$ may be 3 or 4 in the symbol II$(j)$.

We make the definitions $|a_J{}^{\pm}\rangle=|JM+-\rangle\pm|JM-+\rangle$ and $|\beta_J{}^{\pm}\rangle=|JM++\rangle\pm|JM--\rangle$. Since $P=(-1)^{L+1}$, $C=(-1)^{L+S}$ if applicable, and $G=(-1)^{L+S+I}$ for $N\bar{N}$ states, it is clear that these final states, $|a_J{}^{\pm}\rangle$ and $|\beta_J{}^{\pm}\rangle$, are eigenstates of $P$ and $G$ with eigenvalues as follows:

$$P|a_J{}^{\pm}\rangle=\pm(-1)^J|a_J{}^{\pm}\rangle,$$
$$P|\beta_J{}^{\pm}\rangle=\pm(-1)^J|\beta_J{}^{\pm}\rangle,$$
$$G|a_J{}^{\pm}\rangle=\pm(-1)^{J+I}|a_J{}^{\pm}\rangle,$$
$$G|\beta_J{}^{\pm}\rangle=(-1)^{J+I}|\beta_J{}^{\pm}\rangle.$$

For the neutral reactions, II(1) and II(2), they are also eigenstates of $C$:

$$C|a_J{}^{\pm}\rangle=\pm(-1)^J|a_J{}^{\pm}\rangle,$$
$$C|\beta_J{}^{\pm}\rangle=(-1)^J|\beta_J{}^{\pm}\rangle.$$

These results are summarized in Table I for the transition amplitudes leading to final states with odd parity.